Topic 9 –
Matrices of Linear Transformations

Ex: Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 8x-y \end{pmatrix}$$

In the last section we saw that $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ had eigenvalues $\lambda = 3, -1$ and corresponding eigenvectors $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Let $\vec{a} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

So, $T(\vec{a}) = 3\vec{a}$ and $T(\vec{b}) = -\vec{b}$.

One can check that $\vec{a}$ and $\vec{b}$ are linearly independent, so $\beta = [\vec{a}, \vec{b}]$ is a basis for $\mathbb{R}^2$.

Check this out:

Suppose we change coordinate systems to β.

Given a vector $\vec{v}$ in $\mathbb{R}^2$, decompose

$$\vec{v} = c_1 \vec{a} + c_2 \vec{b}.$$

any vector can be written this way.

Ex: $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\vec{v} = 2\vec{a} + \vec{b}$

Then, $c_1 = 2$ $c_2 = 1$

Then,

$$T(\vec{v}) = T(c_1\vec{a} + c_2\vec{b})$$

$$= A(c_1\vec{a} + c_2\vec{b})$$

$$= A(c_1\vec{a}) + A(c_2\vec{b})$$

$$= c_1 A\vec{a} + c_2 A\vec{b}$$

$$= c_1 \cdot 3\vec{a} + c_2(-\vec{b})$$

$$= 3c_1\vec{a} - c_2\vec{b}.$$

$$T(\vec{v}) = T\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix} = 6\vec{a} - \vec{b} = 3\cdot 2\cdot\vec{a} - 1\cdot\vec{b}$$

So, T turns β-coordinates $[\vec{v}]_\beta = \begin{pmatrix}c_1\\c_2\end{pmatrix}$

into β-coordinates $[T(\vec{v})]_\beta = \begin{pmatrix}3c_1\\-c_2\end{pmatrix}$

The matrix that does this is

$$\begin{pmatrix}3 & 0\\0 & -1\end{pmatrix}$$

since

$$\underbrace{\begin{pmatrix}3 & 0\\0 & -1\end{pmatrix}}_{\text{computes } T}\underbrace{\begin{pmatrix}c_1\\c_2\end{pmatrix}}_{\text{but input is } \beta\text{-coordinates}} = \underbrace{\begin{pmatrix}3c_1\\-c_2\end{pmatrix}}_{\text{and output is } \beta\text{-coordinates}}$$

We are now going to develop a way to do this for any basis β and linear transformation T.

Def: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\beta = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$ be a basis for $\mathbb{R}^n$ and γ be a basis for $\mathbb{R}^m$.

The matrix

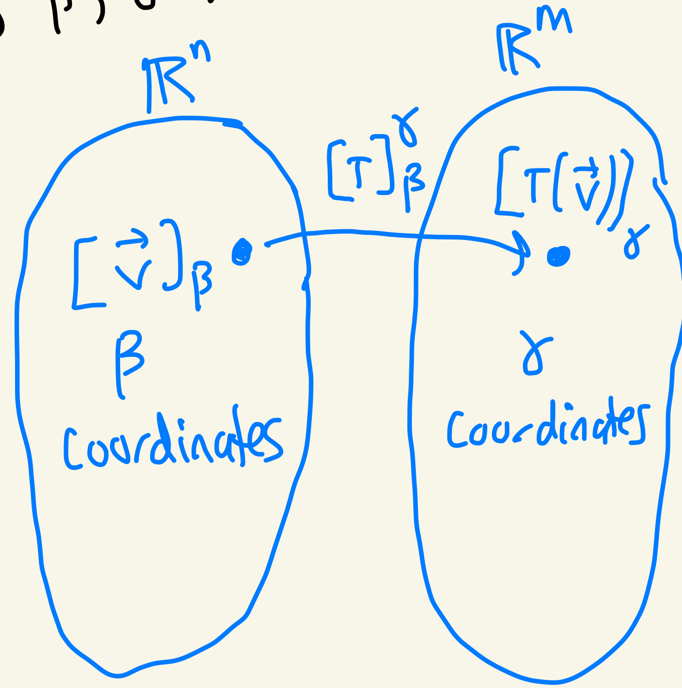
$$[T]_\beta^\gamma = \left([T(v_1)]_\gamma \;\middle|\; [T(v_2)]_\gamma \;\middle|\; \cdots \;\middle|\; [T(v_n)]_\gamma \right)$$

put as columns of matrix

is called the <u>matrix for T with respect to β and γ.</u> If $n=m$ and $\beta=\gamma$, then we just write $[T]_\beta$ for $[T]_\beta^\beta$.

Theorem: With $T: \mathbb{R}^n \to \mathbb{R}^m$, β, γ as above then one has that

$$\underbrace{[T(\vec{v})]_\gamma}_{T(\vec{v})\text{'s } \gamma \text{ coordinates}} = \underbrace{[T]_\beta^\gamma}_{T\text{'s matrix}} \underbrace{[\vec{v}]_\beta}_{\vec{v}\text{'s } \beta \text{ coordinates}}$$



__Ex:__ Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined as above with $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 8x-y \end{pmatrix}$,

and let $\beta = [\vec{a}, \vec{b}]$ where

$\vec{a} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

So, β is a basis for $\mathbb{R}^2$ consisting of eigenvectors of $\mathbb{R}^2$.

Let's compute $[T]_\beta = [T]_\beta^\beta$.

We have

$$T(\vec{a}) = 3\vec{a} = 3\vec{a} + 0\vec{b}$$

$$T(\vec{b}) = -\vec{b} = 0\vec{a} - \vec{b}$$

plug β into T

Write the answer in terms of β

Then,

$$[T]_\beta = \left([T(\vec{a})]_\beta \;\middle|\; [T(\vec{b})]_\beta \right) = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

This is the matrix we got before.

What does it do?

You give it β-coordinates and it computes T but gives you back β coordinates.

For example, if $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ then

$$[T]_\beta [\vec{v}]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 6 \\ -1 \end{pmatrix}}$$

$$\boxed{\begin{gathered} \vec{v} = 2\vec{a} + \vec{b} \\ \text{since} \\ \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{gathered}}$$

this means $[T(\vec{v})]_\beta = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$

ie, $T(\vec{v}) = 6\vec{a} - \vec{b}$

and its true since

$T(\vec{v}) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 6\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Ex:

Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

So, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$.

T is a linear transformation.

Consider the bases $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$ ← $\mathbb{R}^2$ standard basis

and $\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]$.

First we calculate $[T]_\beta^\beta$

$$\underbrace{T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}} = 1\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underbrace{T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+1 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\text{plug } \beta \text{ into } T} \underbrace{= 1\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{express answer as } \beta \text{ coordinates}}$$

So,

$$[T]_\beta^\beta = \left(\left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_\beta \middle| \left[T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_\beta \right) = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Note this is the standard basis matrix for T.
It takes β coordinates as input, computes T and gives you β coordinates as output.

An example is:

Let $\vec{v} = \begin{pmatrix}1\\2\end{pmatrix}$.

Then, $T(\vec{v}) = \begin{pmatrix}1+2\\2-2\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$

Note that $[\vec{v}]_\beta = \begin{pmatrix}1\\2\end{pmatrix}$ since $\vec{v} = 1\cdot\begin{pmatrix}1\\0\end{pmatrix} + 2\cdot\begin{pmatrix}0\\1\end{pmatrix}$.

And $[T(\vec{v})]_\beta = \begin{pmatrix}3\\0\end{pmatrix}$ since $T(\vec{v}) = \begin{pmatrix}3\\0\end{pmatrix} = 3\cdot\begin{pmatrix}1\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\end{pmatrix}$

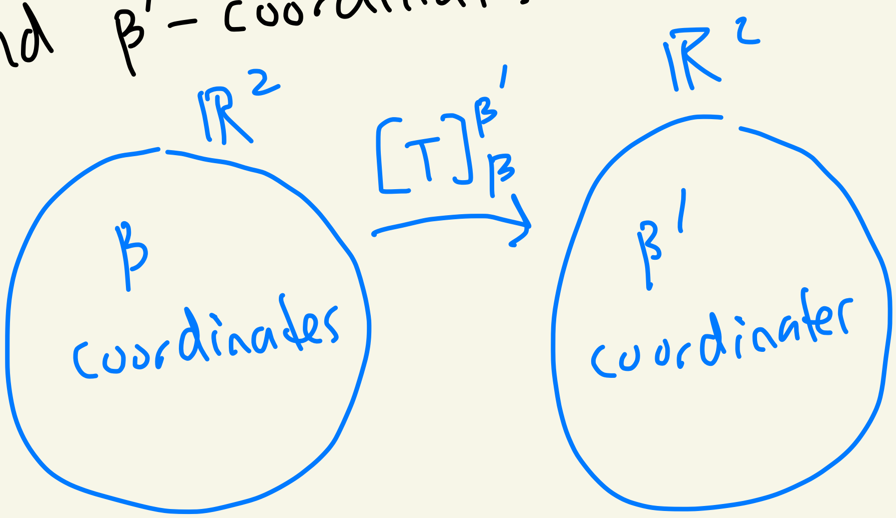
And we have that

$$\underbrace{[T]_\beta^\beta\,[\vec{v}]_\beta}_{\substack{\text{give the}\\ \text{matrix } \beta\\ \text{coordinates}}} = \begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} = \underbrace{\boxed{[T(\vec{v})]_\beta}}_{\substack{T(\vec{v})\text{'s}\\ \beta\text{-coordinates}}}$$

Let's now change bases.

Lets pick β-coordinates for the input to T

and β'-coordinates for the output.



Let's calculate $[T]_\beta^{\beta'}$.

We have

$$\underbrace{T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1+0\\2-0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}}_{\text{plug } \beta \text{ into } T} = a\begin{pmatrix}1\\1\end{pmatrix} + c\begin{pmatrix}-1\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0+1\\0-1\end{pmatrix} = \underbrace{\begin{pmatrix}1\\-1\end{pmatrix} = b\begin{pmatrix}1\\1\end{pmatrix} + d\begin{pmatrix}-1\\1\end{pmatrix}}_{\text{find the } \beta' \text{ coordinates}}$$

Let's find a & c first.

Need to solve:

$$\boxed{\begin{pmatrix}1\\2\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + c\begin{pmatrix}-1\\1\end{pmatrix}} \to \boxed{\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}a-c\\a+c\end{pmatrix}} \to \boxed{\begin{array}{l}a-c=1\\a+c=2\end{array}} \to \boxed{\begin{array}{l}a=3/2\\c=1/2\end{array}}$$

To find b & c we have:

$$\boxed{\begin{pmatrix}1\\-1\end{pmatrix} = b\begin{pmatrix}1\\1\end{pmatrix} + d\begin{pmatrix}-1\\1\end{pmatrix}} \to \boxed{\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}b-d\\b+d\end{pmatrix}} \to \boxed{\begin{array}{l}b-d=1\\b+d=-1\end{array}} \to \boxed{\begin{array}{l}b=0\\d=-1\end{array}}$$

Thus,

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} = \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = 0\begin{pmatrix}1\\1\end{pmatrix} - 1\cdot\begin{pmatrix}-1\\1\end{pmatrix}$$

So,

$$[T]_\beta^{\beta'} = \left(\left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_{\beta'} \middle| \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_{\beta'} \right)$$

$$= \left(\left[\begin{pmatrix}1\\2\end{pmatrix}\right]_{\beta'} \middle| \left[\begin{pmatrix}1\\-1\end{pmatrix}\right]_{\beta'} \right) = \begin{pmatrix} 3/2 & 0 \\ 1/2 & -1 \end{pmatrix}$$

How do we use this?

$[T]_\beta^{\beta'}$ computes T. It takes as inputs β-coordinates and outputs β'-coordinates.

For example, take again $\vec{v} = \begin{pmatrix}1\\2\end{pmatrix}$.

Then $T(\vec{v}) = \begin{pmatrix}1+2\\2-2\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$.

We know that $[\vec{v}]_\beta = \begin{pmatrix}1\\2\end{pmatrix}$ because $\vec{v} = \begin{pmatrix}1\\2\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\end{pmatrix} + 2\cdot\begin{pmatrix}0\\1\end{pmatrix}$

And,

$$[T]_\beta^{\beta'}[\vec{v}]_\beta = \begin{pmatrix} 3/2 & 0 \\ 1/2 & -1 \end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix} 3/2+0 \\ 1/2-2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix}$$

Let's show that $\begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix} = [T(\vec{v})]_{\beta'}$

This is true because

$$\underbrace{\frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{3}{2}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}}_{\text{this demonstrates } \begin{pmatrix}3\\0\end{pmatrix}\text{'s } \beta'\text{-coordinates.}} = T(\vec{v})$$

END OF EXAMPLE

One cool thing about linear transformations is that they show you what matrix products are doing. They are the composition of their corresponding linear transformations.

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and $S: \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations where $T(\vec{v}) = A\vec{v}$ and $S(\vec{w}) = B\vec{w}$ where A is $m \times n$ and B is $k \times m$. Then, $S \circ T$ is a linear transformation and

$$(S \circ T)(\vec{v}) = BA\vec{v}$$

Here $(S \circ T)(\vec{v}) = S(T(\vec{v}))$ is function composition

Proof:

$$(S \circ T)(\vec{v}) = S(T(\vec{v})) = S(A\vec{v}) = BA\vec{v}$$

$\boxtimes$

Ex: Let

$T: \mathbb{R}^2 \to \mathbb{R}^2$ be $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$

$S: \mathbb{R}^2 \to \mathbb{R}^3$ be $S\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ a+b \\ 2a \end{pmatrix}$.

Then, $T\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix}$ and $S\begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}}_{B}\begin{pmatrix} a \\ b \end{pmatrix}$.

We have

$$\begin{aligned}
(S \circ T)\begin{pmatrix} x \\ y \end{pmatrix} &= S\left(T\begin{pmatrix} x \\ y \end{pmatrix}\right) \\
&= S\begin{pmatrix} x+y \\ 2x-y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y+2x-y \\ 2x+2y \end{pmatrix} \\
&= \begin{pmatrix} x+y \\ 3x \\ 2x+2y \end{pmatrix}
\end{aligned}$$

EQUAL

And,

$$\begin{aligned}
BA\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 0 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} x+y \\ 3x \\ 2x+2y \end{pmatrix}
\end{aligned}$$

So,

$$(S \circ T)\begin{pmatrix} x \\ y \end{pmatrix} = BA\begin{pmatrix} x \\ y \end{pmatrix}.$$